Calculus II: For Physical Sciences and Engineering (AS.110.109)

Lecture: 22

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Textbook Sections: 11.8

Todays Topic: Power Series

Today, we will broaden our point of view by considering infinite series made out of functions instead of just real numbers. We start with a foundational definition:

Def: A power series is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots$$

Here, we think of x as a variable. Note that the convergence/divergence of this series depends on the choice of x! So, when thinking of $f(x) = \sum_{n=0}^{\infty} c_n x^n$ as a function, we need to be careful in understanding its domain (all those x so that the series converges).

More generally: We can shift x so that the series is "centered" at some point a:

$$\sum_{n=1}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \cdots$$

Let's jump right into some examples:

1) If $c_n = 1$, $a = 0$, we have $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots = \frac{1}{1-x}$ which has domain $(-1, 1)$.

2) Let $c_n = n!$ What is the domain of convergence for $\sum_{n=0}^{\infty} n! x^n$?

Let's fix some x, and ask what happens if we form the series $\sum_{n=0}^{\infty} a_n$ with $a_n = n! x^n$.

If we apply the ratio test, we compute:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)! x^{n+1}}{n! x^n}\right| = \lim_{n\to\infty} |(n+1)x| = \begin{cases} \infty & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Thus, $\sum_{n=0}^{\infty} n! x^n$ converges iff $x = 0$.

3) $\sum_{n=0}^{\infty} \frac{(x-1)^{3n}}{n}$

Again fix some x. This time, let's try the root test with $a_n = \frac{(x-1)^{3n}}{n}$:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(x-1)^{3(n+1)}}{n+1} \frac{n}{(x-1)^{3n}}\right| = \lim_{n\to\infty} \left|\frac{n}{n+1} \cdot (x-1)^3\right| = |x-1|^3$$

So, if $|x-1| < 1$ (i.e. $0 < x < 2$) the ratio test $\Rightarrow \sum_{n=0}^{\infty} \frac{(x-1)^{3n}}{n}$ converges at x, and if $|x-1| > 1$, the series diverges. At $x = 2$ the series is $\sum_{n=0}^{\infty} \frac{1}{n} = \infty$.

At $x = 0$, the series is $\sum_{n=0}^{\infty} (-1)^{3n} \frac{1}{n}$ which converges.

(Ex: Show the same using the root test: use the fact that $\lim_{n\to\infty} n^{\frac{1}{n}} = \lim_{x\to\infty} e^{\frac{1}{n}\ln n} = 1$.)

Why do we care? As we will introduce soon, power series allow give us useful representations for many ubiquitous functions in math/sciences.

Here are some of the fundamental, general facts:

Theorem: Let $\sum_{n=0}^{\infty} c_n (x-a)^n$ be a power series. Then exactly one of the following occurs:

1) The series converges for every $x \in \mathbb{R}$. In this case, we call $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ an analytic function.
2) The series only converges when $x = a$.
3) There exists a finite "radius of convergence" $R > 0$ so that $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ converges when $|x-a| < R$ and diverges when $|x-a| > R$. $f(x)$ can converge or diverge at $x = a \pm R$!

Example: For the previous examples, we have:

Ex:	Radius of Conv.	Domain of convergence
1)	1	$(-1,1)$
2)	0	$\{0\}$
3)	1	$[0,2)$

More Examples: Pro tip: Usually ratio/root tests are useful for finding R. Use other tests at the endpoints, since the ratio/root tests always fail there.

4) $\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$ $\quad R=\frac{1}{3}$, domain $= (-\frac{1}{3}, \frac{1}{3}]$

Let $a_n = \frac{(-3)^n x^n}{\sqrt{n+1}}$. By the ratio test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-3)^{n+1}}{\sqrt{n+2}}x^{n+1}\frac{\sqrt{n+1}}{(-3)^n x^n}\right| = 3|x|\lim_{n\to\infty}\frac{\sqrt{n+1}}{\sqrt{n+2}} = 3|x|$$

So, $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$ iff $|x| < R = \frac{1}{3}$. When $x=-\frac{1}{3}$, $a_n = \frac{1}{\sqrt{n+1}}$, so $\sum$ diverges.

When $x=\frac{1}{3}$, $a_n = (-1)^n/\sqrt{n+1}$, so $\sum$ converges.

5) Let $J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$ be the 0th order Bessel function (which is just as important as sine/cosine in problems such as heat flow and vibration). $R=\infty$ and domain $= (-\infty, \infty)$.

Let $a_n = \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$. Then by the ratio test: for any $x \in \mathbb{R}$,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^2}{4(n+1)^2}\right| = 0.$$

Thus, $J_0(x)$ is defined on all of $\mathbb{R}$.

6) $\sum_{n=0}^{\infty} \frac{x^n}{3^{n+(-1)^n}}$. $\quad R=1$ $\quad$ domain $= [-1,1]$.

In this case, the ratio test fails: $\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x|}{3}3^{2(-1)^n}$ doesn't have a limit.

Try the root test (in general root test is stronger):

$\sqrt[n]{|a_n|} = |x| \cdot \frac{1}{3^{1+(-1)^n/n}} \to \frac{|x|}{3}$ as $n\to\infty$. So, $R=3$. At $x=\pm 3$, fails $\lim a_n = 0$.